FINITE ELEMENT SOLUTION OF SUBSONIC COMPRESSIBLE FLOW IN AXIAL FLOW TURBOMACHINES

By

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DEPARTMENT OF MECHANICAL ENGINEERING
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CERTIFICATE

Certified that the work entitled "Finite Element Solution of Subsonic Compressible Flow in Axial Flow Turbomachines" which is being submitted by Mr. Romil Kumar Kulshreshtha in partial fulfilment of the requirements for the award of the degree of MASTER OF TECHNOLOGY, has been carried out under our supervision and has not been submitted elsewhere for the award of a degree.

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Romil Kumar Kulshreshtha

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NOMENCLATURE

$\underline{A}$, $\underline{B}$, $\underline{C}$	Coefficients of governing differential equation terms
A_e	Area of the Element
A_{ec}	Value of $\underline{A}$ at the Element Centroid
C	Nondimensionalised velocity
Co	Stagnation velocity
k	Ratio of C_p and C_v
k_e	Element stiffness
U	Chosen Finite Element Approximation
Q	Load Vector
ϕ	Potential Function
ψ	Stream Function

Subscripts

e	Pertaining to a particular element
ne	Pertaining to the nodes of an element
ij	Difference of values of a function at nodes i and j such as $\phi_{ij} = \phi_i - \phi_j$
i	Pertaining to node i
c	Centroid of element

Superscripts

*	Transformed variables

ABSTRACT

Finite Element Method has been applied to the Inverse and Direct Problems of Subsonic steady and compressible flow of inviscid fluid in turbomachines. A recursive scheme for the solution of three versions of Indirect Problem (which have varying degree of non-linearity) and the Direct Problem has been suggested and the related formulations have been obtained. The Inverse Problem is solved on computer using a computer programme which has the capacity to solve all the three versions of the problem and good agreement of results with reported data is found. On the basis of the severe non-linearity of the governing equations for the Inverse Problem and the suitability of the solution scheme for the same, it is hoped that the Direct Problem (formulated in this work) and other similar non-linear Elliptic problems will also find the scheme suitable.

CHAPTER 1

INTRODUCTION

1.1 The Turbomachine Flow Problem

The usefulness of obtaining solutions to turbomachine flows is beyond dispute. The two main problems of interest are

i) the Inverse Problem which seeks the blade geometry for chosen velocity distribution along the blade stream lines and

ii) the Direct Problem which seeks the solution to the flow through given blading and casing geometry.

The solution of real three-dimensional unsteady flow has not yet been obtained even numerically let alone analytically. Approximate solutions have been obtained by patching together solutions in the meridional or hub-shroud plane and in a cylinderical blade to blade surface. The present work is concerned with the numerical solution of the direct and inverse problems for the blade to blade flow.

Until recently, finite difference methods were exclusively used to obtain turbomachine flow solutions. Adequate references* are available in the literature for such methods. The ability of finite element method to deal with difficult boundary conditions with relative ease has attracted workers to apply such methods to the turbomachinery flows. References to such works are given in Section 1.2.

* e.g. Gostelow [16].

The present work attempts to deal with the Inverse and Direct problems in a unified manner. A successful computer programme has been written for the Inverse problem. The programme uses linear triangular elements, generates it's finite element mesh, uses semiband storage mode for storing useful stiffness matrix elements and solves the system of linear simultaneous equations using a version of Gauss elimination algorithm developed for symmetric coefficient matrices stored in semiband mode. On the basis of the mathematical similarity of the equations governing the Direct and the Inverse problem, it has been anticipated that the formulation proposed for the Direct problem may give good results.

1.2 The Finite Element Method and Compressible Flow

The governing equations for compressible flow of fluids are elliplic or hyperbolic depending on whether the flow is subsonic or supersonic. The equations have varying degree of nonlinearity; therefore there is no general method available for the solution of compressible flow problems using the finite element method. Work has been reported in the field of subsonic and supersonic compressible flow by many authors and the general direct problem of fluid flow has been solved using a variety of finite element formulations. As it has always been, more work has been done in the field of subsonic flow than in that of transonic flow because of additional complications involved in the treatment of the latter. Very little work has been reported in the direction of solution of the inverse problem of compressible fluid flow. The present work deals mainly with the solution of subsonic compressible flow in a turbomachine and therefore, in this section, only the subsonic 2-D flow has been discussed.

M.J. O'Carrol and L.A. Morgan [1] have investigated the utility of linear finite elements (Triangular shape) for the analysis of flow in twisted passages and found that faster convergence is achieved compared to finite difference solution for the same accuracy. K. Washizu and M. Ikegawa [2] have solved the integral

equations governing lifting surfaces in flow field successfully and it has been claimed that their method can be extended to other integral equations of interest. H.B. Awbi and J.H. Swannell [3] have investigated the flow round a circular cylinder in a wind tunnel and found good agreement between the finite element and the experimental results. G.F. Carey [4] has used a perturbation expansion to reduce nonlinear flow equations to a sequence of linear problems with positive definite quadratic functionals and nonlinear lower-order nonhomogeneous expressions.

T.J. Chung, J.T. Oden and S.T. Wu [5] have investigated the usefulness of finite element method developed by Aguirre - Ramirez, Oden and Wu [6] in solving the problem of unsteady flow of a rarefied gas through a constant area duct of irregular cross-section. S.T. Shen [7] has explored the compressible flow phenomenon from the view point of incorporating all the numerical requirements through the finite element method for the solution of the problems of interest. B. Davis and J.A. Hendry [8] have suggested the use of a global variational method with trial functions not satisfying the prescribed boundary conditions for the problem of a two-dimensional channel design and their method can find a ready finite element implementation.

1.3 The Inverse Problem

The inverse problem in the context of turbomachines involves the design of blade profile for desired performance. The two most favoured approaches for its solution entail transformation of the governing equations to the potential ($w = \phi + i\Psi$) plane and to the hodograph plane.

Significant work has been reported in the former class by Costello [9], Stanitz [10] and Payne [11]. In this approach, a suitable velocity profile is chosen to obtain the corresponding blade profile. Costello uses the velocity distribution to select a suitable incompressible potential flow around a unit circle and then uses Lin's [12] transformation to transform the incompressible flow into linearized compressible flow about a cascade. Stanitz has used relaxation method and Green's function to solve a combined continuity-irrotationality equation. The use of Green's function in this case has been facilitated by the linearization of the governing equation using tangent gas approximations and the results were computed at the channel walls, the entrance and the exit. The method was improved upon by Payne who used a slightly different dependent variable without changing the nature of Stanitz's linearized equation.

The advantage of using the hodograph plane approach lies in the linear equations which govern the

potential and the stream functions in the hodograph plane. Significant work in this direction has been reported by Cantrell and Fowler [13], Uenishi [14], Cohen [15], and Payne [11]. Uenishi has reported reasonable agreement between theory and experiment for a wide variety of blades. Cohen has used constant surface velocity aerofoils as test cases to show the validity of his method; however, Payne has cautioned against a hasty interpretation of Cohen's results.

CHAPTER 2

FINITE ELEMENT FORMULATION OF INVERSE PROBLEM

The governing equation for the two-dimensional compressible, irrotational, inviscid and steady flow through a channel can be written as follows; [10]:

$$\underline{A} \frac{\partial^2 \ln C}{\partial \phi^2} + \frac{\partial^2 \ln C}{\partial \Psi^2} = \underline{B} \left(\frac{\partial \ln C}{\partial \phi} \right)^2 + \underline{C} \left(\frac{\partial \ln C}{\partial \Psi} \right)^2 \qquad (2.1)$$

where

$$\underline{A} = \left(1 + \left(\frac{k+1}{2}\right) C^2\right) / \left(1 - \left(\frac{k-1}{2}\right) C^2\right)^{(k+1/k-1)}$$

$$\underline{B} = -C^2 \left(1 + \left(\frac{k+1}{2}\right) C^2\right) / \left(1 - \left(\frac{k-1}{2}\right) C^2\right)^{(2k/k-1)}$$

$$\underline{C} = -C^2 / \left(1 - \left(\frac{k-1}{2}\right) C^2\right)$$

The domain of the problem is a rectangle in the $\phi - \Psi$ plane which has been subdivided into three-noded triangular elements. The values of $\underline{A}$, $\underline{B}$, $\underline{C}$, $\frac{\partial \ln C}{\partial \phi}$ and $\frac{\partial \ln C}{\partial \Psi}$ are evaluated at the nodal points using the available known values of C and thus the governing equation is rendered a quasi-poisson equation. To evaluate the quantities $\underline{A}$, $\underline{B}$, $\underline{C}$, $\frac{\partial \ln C}{\partial \phi}$ and $\frac{\partial \ln C}{\partial \Psi}$ at the nodal points, the following shape function is chosen to approximate the behaviour of C in an element:

$$\ln C \simeq U_e = \left[N1 \quad N2 \quad N3 \right] \left[U_i \right]^T \qquad (2.2)$$

where

$$N1 = \left[\Psi_{32} (\emptyset - \emptyset_2) - \emptyset_{32} (\Psi - \Psi_2) \right] / 2 A_e$$

$$N2 = \left[- \Psi_{31} (\emptyset - \emptyset_3) + \emptyset_{31} (\Psi - \Psi_3) \right] / 2 A_e$$

$$N3 = \left[\Psi_{21} (\emptyset - \emptyset_1) - \emptyset_{21} (\Psi - \Psi_1) \right] / 2 A_e$$

and U_i are nodal values of U.

The variational functional corresponding to Poisson equation of this form is given by

$$I = \iint_{A_e} \left[\frac{1}{2} \left\{ \underline{A}_{ec} \left(\frac{\partial U_e}{\partial \emptyset} \right)^2 + \left(\frac{\partial U_e}{\partial \Psi} \right)^2 \right\} + D_{ne} U_e \right] d\emptyset \, d\Psi \qquad (2.3)$$

where $\underline{A}_{ec}$ is the value of $\underline{A}$ at the centroid of element and D_{ne} is the value of right hand side terms of equation (2.1) at the nodes 1, 2 and 3 of the element.

Minimization of the functional I gives us the following stiffness equation:

$$\left[K_e \right] \left[U_{ne} \right]^T = \left[Q_{ne} \right]^T \qquad (2.4)$$

$$\left[k_e \right] = \underline{A}_{ec} \begin{bmatrix} \emptyset_{32}^2 & \emptyset_{32}\,\emptyset_{31} & \emptyset_{32}\,\emptyset_{21} \\ & \emptyset_{31}^2 & \emptyset_{31}\,\emptyset_{21} \\ \text{Sym} & & \emptyset_{21}^2 \end{bmatrix} + \begin{bmatrix} \Psi_{32}^2 & \Psi_{32}.\Psi_{31} & \Psi_{32}.\Psi_{31} \\ & \Psi_{31}^2 & \Psi_{31}.\Psi_{21} \\ \text{Sym.} & & \Psi_{21}^2 \end{bmatrix} \qquad (2.4a)$$

$$\left\{Q_{ne}\right\} = \begin{Bmatrix} D_1\ (\phi_{32}\psi_2 - \psi_{32}\ \phi_2) + (\psi_{32}\iint \phi\ d\phi\ d\psi - \phi_{32}\iint \psi\ d\phi\ d\psi)/A_e \\ D_2\ (\psi_{31}\ \phi_3 - \phi_{31}\psi_3) + (\phi_{31}\iint \psi\ d\phi\ d\psi - \psi_{31}\iint \phi\ d\phi\ d\psi)/A_e \\ D_3\ (\phi_{21}\psi_1 - \psi_{21}\ \phi_1) + (\psi_{21}\iint \phi\ d\phi\ d\psi - \phi_{21}\iint \psi\ d\phi\ d\psi)/A \end{Bmatrix} \quad (2.4b)$$

$$D_i = -\beta_i\left\{1 + (k+1)\beta_i / 2\right\}\alpha_1 / \left\{1 - (k-1)\beta_i/2\right\}^{2k/k-1}$$

$$-\beta_i / \left\{1 - (k-1)\beta_i / 2\right\}\alpha_2 \ ; \quad (2.5)$$

$$\alpha_1 = (\psi_{32}\ U_1 - \psi_{31}\ U_2 + \psi_{21}\ U_3\ )^2 \ ; \quad (2.5a)$$

$$\alpha_2 = (\ \phi_{32}\ U_1 - \phi_{31}\ U_2 + \phi_{21}\ U_3\ )^2 \ ; \quad (2.5b)$$

$$\beta_1 = e^{U_1\ (\phi_{32}\psi_{21} - \psi_{32}\ \phi_{21})\ /\ A_e} \ ;$$

$$\beta_2 = e^{U_2\ (\phi_{32}\psi_{31} - \psi_{32}\ \phi_{31})\ /\ A_e} \ ; \quad (2.5d)$$

$$\beta_3 = e^{U_3\ (\phi_{31}\psi_{21} - \psi_{31}\ \phi_{21})\ /\ A_e} \ ; \quad (2.5e)$$

$$\iint \phi\ d\phi\ d\psi = \phi_c \cdot A_e ; \iint \psi\ d\phi\ d\psi = \psi_c \cdot A_e \ ; \quad (2.6)$$

$$A_{ec} = (1 + 0.5\ (k+1)\ e^{2\ U_c}) / (1 - 0.5\ (k-1)\ e^{2\ U_c}\ )^{(k+1)/(k-1}$$

$$\left.U_c = U_e\ \right| \quad \phi = \phi_c \ ; \psi = \psi_c \ ;$$

and $\phi_c = (\phi_1 + \phi_2 + \phi_3) / 3 \ ; \psi_3 = (\psi_1 + \psi_2 + \psi_3) / 3$

(2.7)

CHAPTER 3

SOLUTION TECHNIQUE

Some variations of the formulation presented in the previous chapter are used to solve the following three versions of the governing equations:

1) $$\frac{\partial^2 \ln C^*}{\partial \phi^{*2}} + \frac{\partial^2 \ln C^*}{\partial \psi^{*2}} = 0 \tag{3.1}$$

where $k = -1$

and

$$C^* = \frac{C}{1 + \sqrt{1 + C^2}}$$

(Tangent gas approximation and transformation used)

2) $$\frac{\partial^2 \ln C}{\partial \phi^2} + \frac{\partial^2 \ln C}{\partial \psi^2} = \frac{C^2}{1 + C^2}\left\{ \left(\frac{\partial \ln C}{\partial \phi}\right)^2 + \left(\frac{\partial \ln C}{\partial \psi}\right)^2 \right\} \tag{3.2}$$

(Only $k = -1$ used)

3) $$\underline{A}\,\frac{\partial^2 \ln C}{\partial \phi^2} + \frac{\partial^2 \ln C}{\partial \psi^2} = \underline{B}\left(\frac{\partial \ln C}{\partial \phi}\right) + \underline{C}\left(\frac{\partial \ln C}{\partial \psi}\right)^2 \tag{3.3}$$

The following steps are taken to solve the above equations:

1. Divison of the domain into triangular elements with aspect ratio nearly equal to one. This is done in the present work by computer (using an automated mesh generation scheme).

2. Calculation of elementwise node numbers, nodal coordinates and value of $\underline{A}$ at the element centroid in case of equation (3.3).

3. Calculation of element stiffness matrix using equation (4.1). The value of $\underline{A}_{ec}$ is taken as unity for the first two cases.

4. Assemblage of the overall stiffness matrix.

5. Calculation of the value of right hand side terms in the case of equations (3.2) and (3.3) and using this as Di to evaluate the load vector. The load vector is null in the case of equation (3.1).

6. Assemblage of the overall load vector.

7. Application of boundary conditions in a suitable fashion. The technique followed* in the present work for incorporating the boundary conditions is as follows:

   i) Multiply the diagonal term of the overall stiffness matrix corresponding to the degree of freedom which has to be restrained by a very large number, say 10^{10}.

   ii) Multiply the prescribed boundary condition with the corresponding modified diagonal element of

*** Refer Zienkiewicz** [17].

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the overall stiffness matrix and replace the respective element of the load vector by this number.

8. Solution of the obtained set of linear simultaneous equations using a suitable scheme. In the present work a Gauss elimination equation solving routine has been developed for symmetric banded coefficient matrices stored in semiband form.

9. Comparison of the currently obtained solution with the previous solution to find if the solution has converged. This is accomplished by comparing the cumulative relative difference of nodal values with chosen permissible error.

10. Stopping the iterative process if convergance is achieved. Otherwise, rename the new solution as old solution and repeat steps two through ten.

<u>Convergence Criteria</u>

As mentioned above, the cumulative relative difference is compared with a chosen permissible error parameter. The cumulative relative difference is calculated as follows:

C.R.I.P. (Cumulative relative improvement parameter)

$$= \sum_{i=1}^{N} \left\| \frac{\ln C_{new} - \ln C_{old}}{\ln C_{new}} \right\|$$

where N is the total number of nodes.

CHAPTER 4

RESULTS AND CONCLUSIONS

The results obtained for versions 1, 2 and 3 are presented in Tables 1, 2 and 3 respectively. Table 4 shows the values of C.R.I.P. and the central processor unit time required per iteration for versions 2 and 3. Figure 1 depicts the solutions of all the three versions.

It is seen that the method used gives almost identical results for versions 1 (Eqn. 3.1) and 2 (Eqn. 3.2) of the inverse problem. It is also seen that the use of the same interpolating function to solve the third version of the problem gives results which compare very well with the results reported by Stanitz [10]*. The solutions of versions 2 (Eqn. 3.2) and 3 (Eqn. 3.3) are found to converge after two iterations only when the convergence criterion is average one percent improvement in the nodal values and the order of magnitude of C.R.I.P becomes unity on the completion of the second iteration.

It is found that no improvement of values is effected by carrying out more than six and seven iterations in the case of versions 2 and 3 respectively and in both cases the order of magnitude of C.R.I.P. becomes 10^{-3} on the completion of the third iteration.

*** Values differ at the most by 0.1%**

After the study of the results obtained, the following is concluded:

1. The same degree of accuracy can be obtained by solving versions 1 and 2 using the same type of elements and interpolating function. This renders the analytical manipulations based on the tangent gas approximation as done by Stanitz [10] for the linearization of the Eqns. (i.e., getting version 1) unnecessary, particularly in view of the very small number of iterations required for the convergence.

2. The version 3 is a better choice in comparison with the versions 1 and 2. The excess computational time needed in the case of the former is not much and also this excess time is more than compensated by the improvement obtained in the nodal values.

3. Further improvement in the values obtained can be effected by using a higher order element and an iterative scheme for solution of linear equations. The latter may require special iterative techniques suited to the semi-band storage of the symmetric and banded stiffness matrix. Another manner in which the improvement can be expected is through the use of quadrilateral element in place of the linear triangular element.

4. As far as the present problem is concerned the convergence requirement of one percent improvement per node (on the average) seems to be quite adequate.

TABLE 1

Φ^* \ Ψ^*	0	1/6	2/6	3/6	4/6	5/6	1
-11/6!	.4009	.4009	.4009	.4009	.4009	.4009	.4009
-10/6!	.4009	.3910	.3914	.3707	.3822	.3976	.4009
-9/6!	.4009	.3815	.3642	.3233	.3659	.3868	.4009
-8/6!	.4009	.3868	.3742	.3523	.3751	.3887	.4009
-7/6!	.4009	.3917	.3840	.3798	.3845	.3923	.4009
-6/6!	.4009	.3952	.3909	.3889	.3910	.3955	.4009
-5/6!	.4009	.3976	.3957	.3942	.3953	.3977	.4009
-4/6!	.4009	.3993	.3980	.3976	.3981	.3993	.4009
-3/6!	.4009	.4005	.4002	.4001	.4003	.4006	.4009
-2/6!	.4009	.4015	.4020	.4024	.4024	.4019	.4009
-1/6!	.4009	.4026	.4041	.4050	.4050	.4037	.4009
0!	.4009	.4040	.4067	.4086	.4090	.4069	.4009
1/6!	.4009	.4058	.4103	.4138	.4155	.4140	.4072
2/6!	.4009	.4081	.4149	.4209	.4253	.4270	.4243
3/6!	.4009	.4108	.4206	.4299	.4382	.4450	.4499
4/6!	.4009	.4138	.4270	.4403	.4533	.4659	.4760
5/6!	.4009	.4171	.4340	.4516	.4698	.4889	.5094
6/6!	.4009	.4205	.4413	.4633	.4869	.5127	.5415
7/6!	.4009	.4239	.4485	.4750	.5041	.5364	.5732
8/6!	.4009	.4271	.4551	.4864	.5207	.5594	.6038
9/6!	.4009	.4302	.4620	.4972	.5366	.5612	.6329
10/6!	.4009	.4330	.4682	.5073	.5514	.6017	.6602
11/6!	.4009	.4355	.4739	.5166	.5650	.6206	.6855
12/6!	.4009	.4378	.4789	.5250	.5774	.6378	.7066
13/6!	.4009	.4399	.4834	.5325	.5885	.6532	.7293
14/6!	.4009	.4417	.4875	.5392	.5983	.6669	.7477
15/6!	.4009	.4434	.4911	.5451	.6069	.6787	.7636
16/6!	.4009	.4449	.4943	.5503	.6144	.6888	.7769
17/6!	.4009	.4463	.4974	.5551	.6209	.6972	.7875
18/6!	.4009	.4480	.5006	.5597	.6267	.7040	.7953
19/6!	.4009	.4502	.5045	.5647	.6322	.7095	.8001
20/6!	.4009	.4540	.5101	.5708	.6380	.7139	.8018
21/6!	.4072	.4622	.5188	.5789	.6447	.7182	.8018
22/6!	.4243	.4769	.5316	.5897	.6529	.7229	.8018
23/6!	.4489	.4971	.5483	.6031	.6627	.7284	.8018
24/6!	.4780	.5212	.5679	.6186	.6738	.7345	.8018
25/6!	.5094	.5474	.5895	.6355	.6858	.7410	.8018
26/6!	.5415	.5746	.6119	.6530	.6982	.7477	.8018
27/6!	.5732	.6017	.6343	.6706	.7106	.7543	.8018
28/6!	.6038	.6231	.6561	.6877	.7226	.7607	.8018
29/6!	.6329	.6531	.6770	.7040	.7340	.7667	.8018
30/6!	.6602	.6767	.6965	.7193	.7446	.7723	.8018
31/6!	.6855	.6985	.7145	.7333	.7543	.7773	.8018
32/6!	.7086	.7183	.7309	.7459	.7631	.7818	.8018

TABLE 1 CONTINUED

ϕ^* \ ψ^*	0	1/6	2/6	3/6	4/6	5/6	1
33/6!	.7293	.7361	.7455	.7572	.7708	.7858	.8018
34/6!	.7477	.7518	.7584	.7671	.7775	.7893	.8018
35/6!	.7636	.7653	.7694	.7756	.7834	.7923	.8018
36/6!	.7769	.7765	.7786	.7827	.7882	.7947	.8018
37/6!	.7875	.7856	.7861	.7894	.7921	.7967	.8018
38/6!	.7953	.7923	.7917	.7928	.7951	.7983	.8018
39/6!	.8001	.7969	.7957	.7960	.7974	.7994	.8018
40/6!	.8018	.7993	.7982	.7982	.7990	.8002	.8018
41/6!	.8018	.8005	.7997	.7996	.8000	.8008	.8018
42/6!	.8018	.8011	.8006	.8005	.8007	.8012	.8018
43/6!	.8018	.8014	.8011	.8010	.8011	.8014	.8018
44/6!	.8018	.8015	.8014	.8013	.8014	.8016	.8018
45/6!	.8018	.8017	.8015	.8015	.8016	.8017	.8018
46/6!	.8018	.8017	.8017	.8016	.8017	.8017	.8018
47/6!	.8018	.8017	.8017	.8017	.8017	.8017	.8018
48/6!	.8018	.8018	.8018	.8017	.8018	.8018	.8018
49/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018
50/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018

TABLE 2

Φ* \ Ψ*	0	1/6	2/6	3/6	4/6	5/6	1
-11/6!	.4009	.4009	.4009	.4009	.4009	.4009	.4009
-10/6!	.4009	.3910	.3914	.3707	.3822	.3926	.4009
-9/6!	.4009	.3815	.3642	.3233	.3658	.3666	.4009
-8/6!	.4009	.3869	.3742	.3673	.3751	.3897	.4009
-7/6!	.4009	.3917	.3840	.3798	.3845	.3923	.4009
-6/6!	.4009	.3952	.3908	.3889	.3910	.3955	.4009
-5/6!	.4009	.3976	.3952	.3942	.3953	.3977	.4009
-4/6!	.4009	.3993	.3980	.3976	.3981	.3993	.4009
-3/6!	.4009	.4005	.4002	.4001	.4003	.4006	.4009
-2/6!	.4009	.4015	.4020	.4024	.4024	.4019	.4009
-1/6!	.4009	.4026	.4041	.4050	.4050	.4037	.4009
0!	.4009	.4040	.4067	.4086	.4090	.4069	.4009
1/6!	.4009	.4056	.4103	.4138	.4155	.4140	.4072
2/6!	.4009	.4081	.4149	.4209	.4253	.4270	.4243
3/6!	.4009	.4108	.4206	.4299	.4382	.4450	.4498
4/6!	.4009	.4138	.4270	.4402	.4533	.4659	.4780
5/6!	.4009	.4171	.4340	.4516	.4698	.4889	.5094
6/6!	.4009	.4205	.4413	.4633	.4869	.5127	.5415
7/6!	.4009	.4239	.4485	.4750	.5041	.5364	.5732
8/6!	.4009	.4271	.4554	.4864	.5207	.5594	.6038
9/6!	.4009	.4301	.4620	.4972	.5365	.5812	.6329
10/6!	.4009	.4330	.4682	.5073	.5513	.6017	.6602
11/6!	.4009	.4355	.4739	.5166	.5650	.6206	.6855
12/6!	.4009	.4376	.4789	.5249	.5773	.6378	.7086
13/6!	.4009	.4399	.4834	.5325	.5884	.6532	.7293
14/6!	.4009	.4417	.4874	.5391	.5983	.6669	.7477
15/6!	.4009	.4434	.4910	.5450	.6069	.6787	.7636
16/6!	.4009	.4449	.4943	.5503	.6143	.6888	.7769
17/6!	.4009	.4463	.4974	.5550	.6209	.6972	.7875
18/6!	.4009	.4480	.5006	.5597	.6267	.7040	.7953
19/6!	.4009	.4502	.5045	.5646	.6322	.7094	.8001
20/6!	.4009	.4540	.5100	.5708	.6379	.7139	.8018
21/6!	.4072	.4622	.5187	.5789	.6447	.7181	.8018
22/6!	.4243	.4769	.5315	.5897	.6529	.7229	.8018
23/6!	.4489	.4971	.5482	.6031	.6627	.7284	.8018
24/6!	.4780	.5211	.5679	.6186	.6738	.7345	.8018
25/6!	.5094	.5474	.5895	.6355	.6858	.7410	.8018
26/6!	.5415	.5746	.6118	.6530	.6982	.7477	.8018
27/6!	.5732	.6017	.6343	.6706	.7106	.7543	.8018
28/6!	.6038	.6281	.6561	.6877	.7226	.7607	.8018
29/6!	.6329	.6531	.6770	.7040	.7340	.7667	.8018
30/6!	.6602	.6767	.6965	.7193	.7446	.7723	.8018
31/6!	.6855	.6985	.7145	.7333	.7543	.7773	.8018
32/6!	.7086	.7183	.7309	.7459	.7631	.7818	.8018
33/6!	.7293	.7361	.7455	.7572	.7708	.7858	.8018

TABLE 2 CONTINUED

Φ^* \ ψ^*	0	1/6	2/6	3/6	4/6	5/6	1
34/6!	.7477	.7518	.7584	.7671	.7776	.7893	.8018
35/6!	.7636	.7653	.7694	.7756	.7834	.7923	.8018
36/6!	.7769	.7765	.7786	.7827	.7882	.7947	.8018
37/6!	.7875	.7856	.7861	.7884	.7921	.7967	.8018
38/6!	.7953	.7923	.7917	.7928	.7951	.7983	.8018
39/6!	.8001	.7969	.7957	.7960	.7974	.7994	.8018
40/6!	.8018	.7993	.7982	.7982	.7990	.8002	.8018
41/6!	.8018	.8005	.7997	.7996	.8000	.8008	.8018
42/6!	.8018	.8011	.8006	.8005	.8007	.8012	.8018
43/6!	.8018	.8014	.8011	.8010	.8011	.8014	.8018
44/6!	.8018	.8015	.8014	.8013	.8014	.8016	.8018
45/6!	.8018	.8017	.8015	.8015	.8016	.8017	.8018
46/6!	.8018	.8017	.8017	.8016	.8017	.8017	.8018
47/6!	.8018	.8017	.8017	.8017	.8017	.8017	.8018
48/6!	.8018	.8018	.8018	.8017	.8018	.8018	.8018
49/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018
50/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018

[illegible] 3

Θ^* \ Ψ^*	0	1/6	2/6	3/6	4/6	5/6	1
-11/6!	.4009	.4009	[illegible]	[illegible]	[illegible]	[illegible]	.4009
-10/6!	.4009	.4005	.4031	.3399	.4001	.4095	.4009
-9/6!	.4009	.3905	.3979	.3458	.3979	.3935	.4009
-8/6!	.4009	.4004	.4011	.3995	.4001	.4034	.4009
-7/6!	.4009	.4003	.4017	[illegible]	.4007	.4034	.4009
-6/6!	.4009	.4004	.4009	.4006	.4010	.4039	.4009
-5/6!	.4009	.4004	.4009	.4001	.4010	.4039	.4009
-4/6!	.4009	.4004	.4010	.4005	.4009	.4009	.4009
-3/6!	.4009	.4009	.4010	.4003	.4009	.4009	.4009
-2/6!	.4009	.4010	.4010	.4010	.4010	.4010	.4009
-1/6!	.4009	.4011	.4012	.4013	.4013	.4011	.4009
0!	.4009	.4016	.4021	.4024	.4024	.4020	.4009
1/6!	.4009	.4029	.4047	.4063	.4074	.4078	.4072
2/6!	.4009	.4054	.4099	.4141	.4181	.4215	.4243
3/6!	.4009	.4089	.4170	.4252	.4334	.4416	.4498
4/6!	.4009	.4129	.4252	.4379	.4509	.4643	.4780
5/6!	.4009	.4170	.4338	.4514	.4693	.4871	.5094
6/6!	.4009	.4211	.4423	.4649	.4882	.5143	.5415
7/6!	.4009	.4249	.4505	.4779	.5073	.5390	.5732
8/6!	.4009	.4285	.4581	.4901	.5248	.5626	.6038
9/6!	.4009	.4317	.4651	.5014	.5412	.5848	.6329
10/6!	.4009	.4346	.4714	.5118	.5562	.6055	.6602
11/6!	.4009	.4372	.4770	.5211	.5700	.6244	.6855
12/6!	.4009	.4394	.4821	.5294	.5823	.6416	.7086
13/6!	.4009	.4414	.4864	.5367	.5932	.6569	.7293
14/6!	.4009	.4431	.4902	.5431	.6027	.6704	.7477
15/6!	.4009	.4445	.4934	.5485	.6109	.6820	.7636
16/6!	.4009	.4457	.4961	.5530	.6176	.6916	.7769
17/6!	.4009	.4466	.4982	.5565	.6230	.6992	.7875
18/6!	.4009	.4473	.4998	.5592	.6270	.7049	.7953
19/6!	.4009	.4490	.5011	.5612	.6298	.7086	.8001
20/6!	.4009	.4491	.5030	.5636	.6323	.7109	.8018
21/6!	.4072	.4557	.5094	.5694	.6369	.7136	.8018
22/6!	.4243	.4712	.5230	.5808	.6455	.7185	.8018
23/6!	.4489	.4933	.5424	.5968	.6573	.7252	.8018
24/6!	.4780	.5196	.5652	.6155	.6711	.7329	.8018
25/6!	.5094	.5478	.5897	.6355	.6857	.7409	.8018
26/6!	.5415	.5765	.6144	.6556	.7002	.7488	.8018
27/6!	.5732	.6046	.6385	.6749	.7141	.7563	.8018
28/6!	.6038	.6317	.6614	.6932	.7271	.7633	.8018
29/6!	.6329	.6571	.6829	.7102	.7390	.7696	.8018
30/6!	.6602	.6809	.7027	.7257	.7499	.7753	.8018
31/6!	.6855	.7027	.7209	.7398	.7597	.7803	.8018
32/6!	.7086	.7226	.7372	.7524	.7683	.7848	.8018

TABLE 3 (CONTINUED)

φ \ ψ*	0	1/6	2/6	3/6	4/6	5/6	1
33/6!	.7243	.7402	.7517	.7630	.7750	.7887	.8018
34/6!	.7377	.7555	.7644	.7773	.7825	.7921	.8018
35/6!	.7546	.7693	.7753	.7816	.7861	.7949	.8018
36/6!	.7750	.7804	.7843	.7884	.7927	.7972	.8018
37/6!	.7875	.7893	.7911	.7934	.7963	.7990	.8018
38/6!	.7953	.7954	.7966	.7977	.7989	.8004	.8018
39/6!	.8001	.7994	.7989	.8002	.8006	.8012	.8018
40/6!	.8018	.8015	.8013	.8015	.8011	.8016	.8018
41/6!	.8018	.8017	.8017	.8017	.8017	.8017	.8018
42/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018
43/6!	.8018	.8018	.8018	.8018	.8018	.8016	.8018
44/6!	.8018	.8018	.8018	.8018	.8017	.8018	.8018
45/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018
46/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018
47/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018
48/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018
49/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018
50/6!	.8018	.8018	.8018	.8018	.8018	.8018	.8018

TABLE 4

ITERATION NO.	VERSION TWO		VERSION THREE	
	C.R.I.P	C.P.U	C.R.I.P	C.P.U
1	.30000E 03	2.13 Sec	.30000E 03	2.28 Sec
2	.58882E 00	4.11 Sec	.24683E 00	4.44 Sec
3	.19621E -4	6.07 Sec	.31854E -3	6.61 Sec
4	.59805E -8	8.05 Sec	.37087E -6	8.77 Sec
5	.10457E -11	10.02 Sec	.10964E -8	10.95 Sec
6	.43506E -12	11.97 Sec	.33832E -11	13.10 Sec
7	-	-	.42862E -12	15.27 Sec

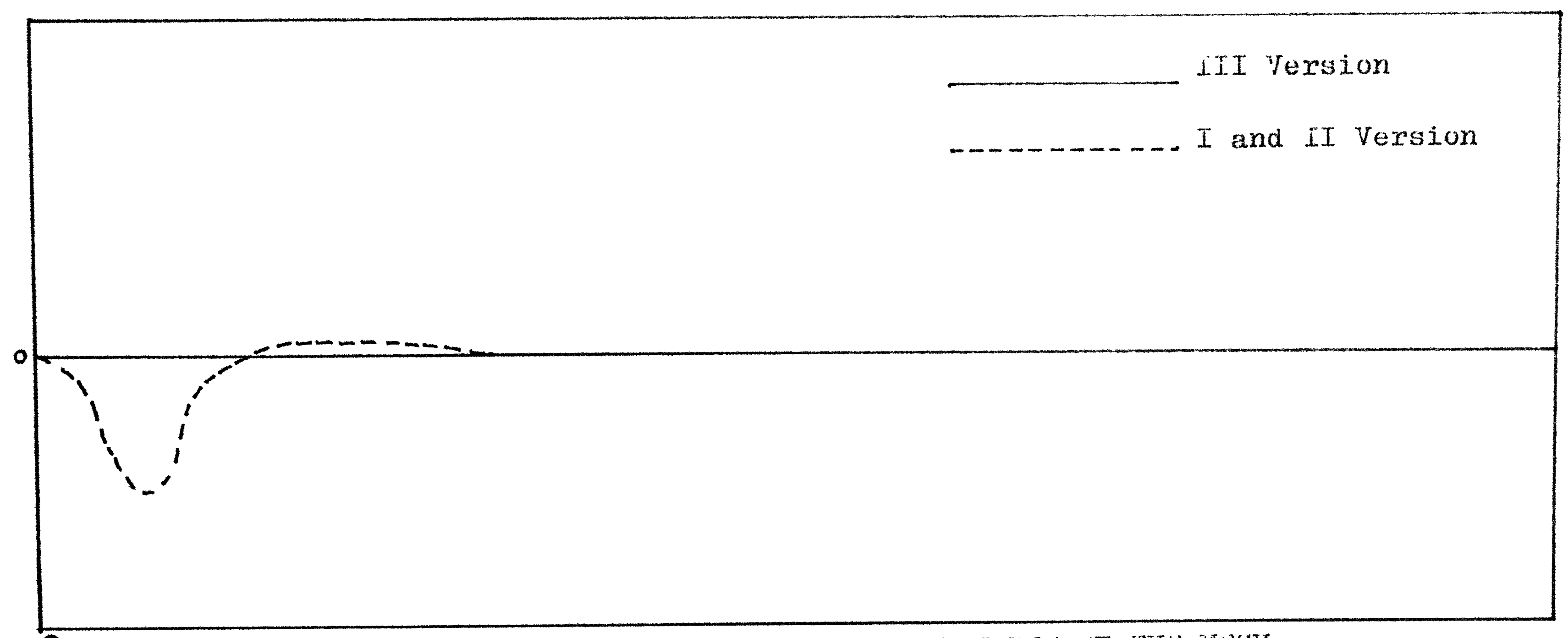

FIGURE 1. CUMOLATIVE RELATIVE ERROR AVERAGED OVER 10 NODES OF THE MESH

SCALE: X Axis : 1cm. equals 10 nodes; Y Axis ; 1cm. equals .0001

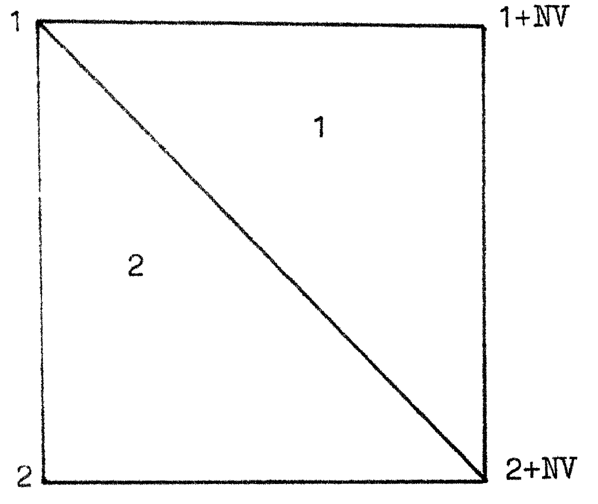

FIGURE 2. DISCRETISATION SCHEME

REFERENCES

1. M.J. O'Carroll and L.A. Morgan, "Difference and Element Methods for Potential Flow", Finite Element Methods in Flow Problems, UAH Press, The University of Albama in Huntsville, Huntsville, Albama, 1974.

2. K. Washizu and M. Ikegava, "Finite Element Technique in Lifting Surface Problems", Finite Element Methods in Flow Problems, UAH Press, The University of Albama in Huntsville, Huntsville, Albama, 1974.

3. H.B. Awbi and J.H. Swammell, "A Finite Element Solution of the Confined Flow Over a Circular Cylinder", Finite Element Methods in Flow Problems, UAH Press, The University of Albama in Huntsville, Huntsville, Albama, 1974.

4. G.F. Carey, "Perturbation Variation Solution of Compressible Flows", Finite Element Methods in Flow Problems, UAH Press, The University of Albama in Huntsville, Huntsville, Albama, 1974.

5. T.J. Chung, J.T. Oden and S.T. Wu, "The Finite Element Analysis of Transient Rarefied Gas Flow," Finite Element Methods in Flow Problems, UAH Press, The University of Albama in Huntsville, Huntsville, Albama, 1974.

6. G. Aguirre - Remirez, J.T. Oden and S.T. Wu, "Advances in Mechanics", Proceedings of the 7th International Symposium of Rarefied Gas Dynamics, AP, NY, 1969.

7. S.F. Shen, "An Aerodynamicist Looks at the Finite Element Method", Finite Element Methods in Flow Problems, UAH Press, The University of Albama in Huntsville, Huntsville, Albama, 1974.

8. P. Davis and J.A. Hendry, "The Indirect Problem of Fluid Channels Using a Global Variational Method With Trial Functions Not Satisfying the Prescribed Boundary Conditions", Int. J. for Num. Meth. in Engg., 11, 579-591, (1977).

9. G.R. Costello, "Method of Designing Cascade Blades With Prescribed Velocity Distributions in Compressible Potential Flow," N.A.C.A. Report 978, 1949.

10. J.D. Stanitz, "Design of Two-Dimensional Channels With Prescribed Velocity Distributions Along the Channel Walls, Part I, Relaxation Solutions," N.A.C.A. T.N. 2593, 1952.

11. D. Payne, "Isolated Cascade Aerofoils," Master's Dissertation, London Univ., 1964.

12. C.C. Lin, "On the Subsonic Flow Through Circular and Straight Lines of Airfoils, "Journal of Mathematics and Physics, Vol. 38, No. 2, 1949, p. 117.

13. H.N. Cantrell and J.E. Fowler, "The Aerodynamic Design of Two-Dimensional Turbine Cascades for Incompressible Flow With a High Speed Computer," J. of Basic Engg., Trans. A.S.M.E., Series D, Vol. 81, No. 3, Sept. 1959, p. 349.

14. A. Uenishi, "A Design Method and the Performance of Two-Dimensional Turbine Cascades for High Subsonic Flow," ASME Paper No. 71 - GT - 34, 1971.

15. M. . Cohen, "A Hodograph Design Method for Compressible Flow Problems," J. of Appl. Mech., Vol. 29, 1962.

16. **Gostelow, "Review of Compressible Flow Theories for Airfoil Cascades," ASME Paper No. 73-GT-9.**

17. **Zienkiewics, "The Finite Element Method", III Ed. Mc Graw Hill, London, 1977, pp. 11.**

APPENDICES

APPENDIX A

A.1 The Direct Problem

The velocity potential function ϕ satisfies the following partial differential equation for two-dimensional, steady, inviscid compressible flows:

$$\phi_{xx} + \phi_{yy} = \frac{1}{C^2} \left[\phi_x^2 \phi_{xx} + \phi_y^2 \phi_{yy} + 2 \phi_x \phi_y \phi_{xy} \right]$$

where

$$C^2 = Co^2 - \frac{1}{2} (k - 1) (\phi_x^2 + \phi_y^2) \qquad (A.1.1)$$

There is no universally valid technique available as yet to solve the above equation. Gostellow has reviewed the various classes of this problem. In this reference, efforts have been directed towards the performance analysis of medium solidity cascades termed as the general direct problem. There are four main categories of solution; namely, series solution methods, iterative methods, matrix methods and streamline curvature methods.

Two classes of work utilizing series solution methods have been reported out of which one treats the velocity potential function as the dependent variable while the other uses the streamline function for the same purpose. The dependent variable is approximated by a series of the following type

$$F = F_0 + F_1 M^2 + F_2 M^4 + \dots + F_n M^{2n} \qquad (A.1.2)$$

where F is the dependent variable $\emptyset$ or Ψ and M is any reference Mach number, F_o is the incompressible solution and F_1, F_2, ... are functions of the lower order coefficients of F and x and y are the independent variables.

The simulation of the right hand side of the governing equation by a source - sink distribution within the field of incompressible flow around the profile and the calculation thereof in iterative steps is the essence of the iterative methods which are reported to converge slowly as the sonic condition is approached.

In most matrix methods, the governing equation is solved as if it were a poisson equation using ten-point-star finite difference scheme or relaxation. The various formulations differ in their treatment of density gradients.

The streamline curvature methods use streamlines and 'Quasi-orthogonals' (lines passing from one channel wall to the other). The input data is differentiated twice and the velocities obtained are then used in the velocity gradient equation. A start is made with approximate velocities, along the midstream line obtained from the previous iteration. A numerical integration is then performed in both directions to find the blade surface velocities and then the velocity levels are adjusted to satisfy the continuity equation.

The finite element methods have also been applied to [illegible] their brief description appears in Section [illegible].

A.2. [illegible] and Solution Technique

The governing equation is rewritten as follows:

$$\nabla^2 \phi = L$$

where

$$L = \frac{1}{c^2} \left[\phi_x^2 \, \phi_{xx} + \phi_y^2 \, \phi_{yy} + 2 \, \phi_x \, \phi_y \, \phi_{xy} \right]$$

and

$$c^2 = Co^2 - \frac{1}{2} (k - 1) (\phi_x^2 + \phi_y^2) \qquad \text{(A.2.1)}$$

The flow field around the profile is divided into straight sided triangular elements and the behaviour of the function ϕ is approximated with a quintic polynomial in the area coordinates Z_1, Z_2 and Z_3:

$$\phi_c = \left\{ f \, (Z_1, \, Z_2) \right\}^T \left\{ a_i \right\} \qquad \text{(A.2.2)}$$

where a_i are coefficients of the terms of the quintic polynomial which are evaluated in terms of the nodal values of the degrees of freedom as follows:

$$\left\{ a_i \right\} = \left[B \right]^{-1} \left\{ \phi_i \right\}_{Z_1, Z_2} \qquad \text{(A.2.3)}$$

The values of ϕ as referred to the physical $x - y$ coordinate system can be evaluated from $\left\{ \phi_i \right\}_{Z_1, Z_2}$ in the following manner:

$$\left\{\phi_i\right\}_{Z_1,Z_2} = \left[T_c\right]\left\{\phi_i\right\}_{xy} \tag{A.2.4}$$

$$\therefore \quad \phi_e = \left\{f\ (Z_1,\ Z_2)\right\}^T \left[B^{-1}\right]\left[T_c\right]\left\{\phi_i\right\}_{xy} \tag{A.2.5}$$

$$= \left\{N_i\right\}^T \left\{\phi_i\right\}_{xy} \tag{A.2.6}$$

As before, the value of L for an element is computed from the known nodal values of ϕ and the usual variational-finite element technique is used to get the stiffness equation as follows:

$$\left[k_e\right]\left\{\phi_i\right\}_{Z_1,Z_2} = \left\{Q_i\right\}_{Z_1,Z_2} \tag{A.2.7}$$

where

$$\left\{\phi_i\right\} = \left\{\phi_1,\ \frac{\partial \phi_1}{\partial Z_1},\ \frac{\partial \phi_1}{\partial Z_2},\ \frac{\partial^2 \phi_2}{\partial Z_1^2},\ \frac{\partial^2 \phi_1}{\partial Z_1\ Z_2},\ \frac{\partial^2 \phi_1}{\partial Z_2^2},\right.$$
$$\phi_2,\ \frac{\partial \phi_2}{\partial Z_1},\ \frac{\partial \phi_2}{\partial Z_2},\ \frac{\partial^2 \phi_2}{\partial Z_1^2},\ \frac{\partial^2 \phi_2}{\partial Z_1\ Z_2},\ \frac{\partial^2 \phi_2}{\partial Z_2^2},$$
$$\left.\phi_3,\ \frac{\partial \phi_3}{\partial Z_1},\ \frac{\partial \phi_3}{\partial Z_3},\ \frac{\partial^2 \phi_3}{\partial Z_1^2},\ \frac{\partial^2 \phi_3}{\partial Z_1\ Z_2},\ \frac{\partial^2 \phi_3}{\partial Z_2^2},\phi_4\right\}^T \tag{A.2.8}$$

$$= \left\{\lfloor \phi_{1k} \rfloor,\ \lfloor \phi_{2k} \rfloor,\ \lfloor \phi_{3k} \rfloor,\ \phi_4\right\}^T ; \quad k = 1,\ 2,\ 3,\ 4\ \&\ 5 \tag{A.2.9}$$

here subscripts 1, 2, 3 and 4 refer to the three nodes of the straight sided triangle and its centroid respectively.

$$C_i = - L_e \iint_{A_e} F(Z_1 , Z_2) \; dA \qquad (a.2.10)$$

$$\iint_{A_e} Z_1^l \; Z_2^m \; Z_3^n \; dA = \frac{2 A_e \; . n ! \; m ! \; l !}{(l + m + n + 2) !} \qquad (A.2.11)$$

$$\left[K \right] = \left[\iint_{A_e} \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right]^T \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right] dA \right] \left[B^{-1} \right]$$

$$= \left[K' \right] \left[B^{-1} \right] \qquad (A.2.12)$$

where

$$\left[K' \right] = \iint_{A_e} \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right]^T \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right] dA$$

$$= \iint_{A_e} \left[\frac{\partial f}{\partial Z_1} \quad \frac{\partial f}{\partial Z_2} \right]^T \left[J^{-1} \right]^T \left[J^{-1} \right] \left[\frac{\partial f}{\partial Z_1} \quad \frac{\partial f}{\partial Z_2} \right] dA$$

(A.2.13)

where J is the well known Jacobian matrix and

$$J^{-1} = \frac{1}{2 A_e} \begin{bmatrix} y_{23} & y_{31} \\ x_{32} & x_{13} \end{bmatrix} \qquad (A.2.14)$$

$$\left\{ \emptyset_{\dot{j}\,\dot{k}} \right\}_{Z_1 , Z_2} = \left[T \right] \left\{ \emptyset_{\dot{j}\,\dot{k}} \right\}_{x,y} \qquad (A.2.15)$$

hwere $\dot{j} = 1, 2, 3$; $\dot{k} = 1, 2, 3, 4, 5$

$$T = \begin{bmatrix} x_{13} & y_{13} & 0 & 0 & 0 \\ x_{23} & y_{23} & 0 & 0 & 0 \\ 0 & 0 & x_{13}^2 & 2 \, x_{13} \, y_{13} & y_{13}^2 \\ 0 & 0 & x_{13} \, x_{23} & \left(\begin{matrix} y_{13} \, x_{23} + \\ x_{13} \, y_{23} \end{matrix} \right) & y_{13} \, y_{23} \\ 0 & 0 & x_{23}^2 & 2 \, y_{23} \, x_{23} & y_{23} \end{bmatrix} \qquad (A.2.16)$$

so that

$$\left\{\phi_i\right\}_{Z_1,Z_2} = \begin{bmatrix} {}^1\left[T\right]_1 & & 0 \\ & \left[T\right]_1 & \\ 0 & & \left[T\right]_1 \end{bmatrix} \left\{\phi_i\right\}_{x_1,x_2}$$

$$= \left[T_e\right] \left\{\phi_i\right\}_{x_1,x_2}$$

The ϕ_4 term is suppressed soon after computation of element stiffness matrix.

$$f\ (Z_1,\ Z_2,\ Z_3) = f\ (Z_1,\ Z_2)$$

$$= 1,\ Z_1,\ Z_2,\ Z_1\ Z_2,\ Z_2\ Z_3,\ Z_1\ Z_3,\ Z_1^2\ Z_2 - Z_2^2\ Z_1,\ Z_2^2\ Z_3 - Z_3^2\ Z_2,\ Z_1^2\ Z_3 - Z_3^2\ Z_1,\ Z_1\ Z_2\ Z_3,\ Z_1^3\ Z_2 - Z_1\ Z_2^3,\ Z_2^3\ Z_3 - Z_2\ Z_3^3,\ Z_1^3\ Z_3 - Z_1\ Z_3^3,\ Z_1^2\ Z_2^2,\ Z_2^2\ Z_3^2,\ Z_1^2\ Z_3^2,\ Z_1^3\ Z_2^2 - Z_1^2\ Z_2^3,\ Z_2^3\ Z_3^2 - Z_2^2\ Z_3^2,\ Z_1^3\ Z_3^2 - Z_1^2\ Z_3^3 \qquad (A.2.18)$$

Solution Technique

The quasi-Poisson equation (A.2.1) is solved iteratively using a suitable method for the solution of the generated set of linear simultaneous equations in the following manner:

$$\left[K\right]\left\{\phi\right\}^{m+1} = \left\{Q\right\}^{m} \qquad (A.3.1)$$

Steps taken to effect this are as follows:

1. Computation of element stiffness matrices.
2. Assembly of overall stiffness matrix and its storage in the disk.
3. Calculation of element load vector Q_e on the basis of the current values of $\emptyset_{ne}$ & its assembly.
4. Retrieval of overall stiffness matrix from disk for iteration numbers greater than one and/or application of the boundary conditions besides enforcing the equality of boundary conditions wherever prescribed.
5. Solution of the system of linear simultaneous equations.
6. Testing of the new solution using a suitable convergence criterion.
7. Stopping the process if the convergence criterion are fulfilled otherwise repetition of steps 3 through 7.

Convergence Criterion

The convergence parameter is compared with the cumulative relative improvement of nodal values in the same way as has been done in the case of the inverse problem.

APPENDIX B

PROGRAMME LISTING AND USERS' INSTRUCTIONS

B.1 Users' Instructions

This programme has been developed for running on DEC 10 Time sharing system and it uses the disk storage for retrieving and filing data. The various inputs to the programme have the following format and significance:

1. NP — An integer number. It controls the flow of computations in respect of different versions of the non-linear problem.

   NP = 1 Linearised Problem (Version 1)

   NP = 2 Partially Linearised Problem (Version 2)

   NP = 3 Complete Non-linear Problem (Version 3)

2. NIT — Do loop parameter which fixes the upper limit on the number of iterations.

   NH — Number of nodes on the horizontal side of the domain.

   NV — Number of nodes on the vertical side of the domain. NH is greater than NV if not equal to the latter.

   PHII & PSIF — Range of values of independent variable represented on the vertical side.

DEL	Damping factor. Taken unity if damping is not needed.
N	Total number of Boundary nodes. Calculated by the Computer itself.
Q(I), I=1,N	Vector of length N. Having the values of the prescribed Boundary conditions as it's elements. Values are read in the following way: 1. Nodal values at the nodes on the left hand side 2. Nodal values at the nodes on the right hand side 3. Top side nodes leaving those already accounted for from left to right. 4. Bottom side nodes in the same manner as in case of 3 above.
XP	The limiting permissible cumulative relative error (In subroutine CONVER)

The data is read from a file named DAT4 CDR from disk.

SUBROUTINES

MAIN 1 and MAIN 2	Compute the nodal numbers and coordinates.
MAIN 3	Compute $\underline{A}_{ec}$ for NP = 3 (i.e., Version 3).
PLACE	Places the boundary conditions appropriately from vector Q read in the main programme

into the vector PHI and generates a vector NO of length N whose elements indicate the number of the nodes on the boundary.

SOLVE	Uses a modified Gauss Elimination algorithm suitable for symmetric matrices stored in semi-band mode to solve the stiffness equation.
CONVER	Tests convergence and renames the new solution as the old solution before returning to the main programme.
WRITE	Writes the results in a file named DESIGN.DAT and stores in users' area.

```
C*************************************************************************
C                  FORTRAN-10 PROGRAM FOR INVERSE PROBLEM
C*************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/B1/PHI(134),Q(434),OK(434,9),NU(135),NNO(3)
      COMMON/B2/NH,NV,NN,NE,NVE,ILT,N,IROW,ICOL,DX,DY,XC,YC,BK(3,3),X(3)
     1,Y(3),Q1(3),B(3),DEL
      ACCEPT *,NP,NIT
      OPEN(UNIT=1,FILE='DAT4.CDR',ACCESS='SEQIN')
      READ(1,*)NH,NV,PHII,PHIF,PSII,PSIF,DEL
      N=2*(NH+NV-2);NN=NH*NV;NVE=2*(NV-1);NE=NVE*(NH-1);ILT=NV+2
      DX=(PHIF-PHII)/FLOAT(NH-1);DY=(PSIF-PSII)/FLOAT(NV-1)
      AR=.5*DX*DY;A4=4.*AR;READ(1,1)(Q(I),I=1,N);CALL PLACE
    1 FORMAT(8(E12.5,3X));CLOSE(UNIT=1,FILE='DAT4.CDR',DISPOSE='SAVE')
      DO 100 ITER=1,NIT
    0 FORMAT(40X,' ITERATION NUMBER ',I5,40(1H*))
      DO 25 I=1,NN;Q(I)=0.0;DO 25 J=1,ILT;OK(I,J)=0.0
   25 CONTINUE
      DO 11 II=1,NE;I=II;IROW=I/NVE+1;ICOL=(I-(I/NVE)*NVE)/2+1;IT=1
      IF((I/2)*2.EQ.I)GOTO 2;GOTO 3
    2 ICOL=ICOL-1;IF((I/NVE)*NVE.EQ.I)ICOL=NV-1
      IF((I/NVE)*NVE.EQ.I)IROW=IROW-1;IT=2
    3 CALL MAIN1(I,IT)
      CALL MAIN2(I,IT,PHII,PSIF)
      XC=(X(1)+X(2)+X(3))/3.;YC=(Y(1)+Y(2)+Y(3))/3.
      J=NNO(1);K=NNO(2);L=NNO(3);A1=PHI(J);B1=PHI(K);C1=PHI(L)
      AL=1.0
      IF(NP.EQ.3)CALL MAIN3(A1,B1,C1,AL,XC,YC)
      GOTO(7,8)IT
    7 BK(1,1)=DY*DY;BK(1,2)=0.;BK(1,3)=-BK(1,1);BK(2,2)=AL*DX*DX
      BK(2,3)=-BK(2,2);BK(3,3)=BK(2,2)+BK(1,1);GOTO 9
    8 BK(1,1)=AL*DX*DX;BK(1,2)=-BK(1,1);BK(1,3)=0.0;BK(2,3)=-DY*DY
      BK(2,2)=BK(1,1)-BK(2,3);BK(3,3)=-BK(2,3)
    9 BK(2,1)=BK(1,2);BK(3,1)=BK(1,3);BK(3,2)=BK(2,3)
      IF(NP.EQ.1)GOTO 110
      AS1=((Y(3)-Y(2))*A1-(Y(3)-Y(1))*B1+(Y(2)-Y(1))*C1)**2
      AS2=((X(3)-X(2))*A1-(X(3)-X(1))*B1+(X(2)-X(1))*C1)**2
      B(1)=EXP(A1*((X(3)-X(2))*(Y(2)-Y(1))-(Y(3)-Y(2))*(X(2)-X(1)))/AR)
      B(2)=EXP(B1*((X(3)-X(1))*(Y(3)-Y(1))-(Y(3)-Y(2))*(X(3)-X(1)))/AR)
      B(3)=EXP(C1*((X(3)-X(1))*(Y(2)-Y(1))-(Y(3)-Y(1))*(X(2)-X(1)))/AR)
      DO 4 IS=1,3;E=-((B(IS)*(1.+1.2*B(IS)))/((1.-.2*B(IS))**7))
      F=-B(IS)/(1.-.2*B(IS));B(IS)=E*AS1+F*AS2
      IF(NP.EQ.2)B(IS)=AS2*(E+F)
    4 CONTINUE
      Q1(1)=B(1)*.5*((X(3)-X(2))*(Y(2)-YC)-(Y(3)-Y(2))*(X(2)-XC))
      Q1(2)=B(2)*.5*((Y(3)-Y(1))*(X(3)-XC)-(X(3)-X(1))*(Y(3)-YC))
      Q1(3)=B(3)*.5*((X(2)-X(1))*(Y(1)-YC)-(Y(2)-Y(1))*(X(1)-XC))
  110 DO 5 IS=1,3;IM=NNO(IS);IF(NP.NE.1)Q(IM)=Q(IM)+Q1(IS);DO 5 JS=1,3
      IN=NNO(JS)-IM+1;IF(IN.LE.0)GOTO 5;OK(IM,IN)=OK(IM,IN)+BK(IS,JS)/A4
    5 CONTINUE
```

```
11    CONTINUE
      DO 6 I=1,N;K=NO(I);OK(K,1)=OK(K,1)*(10.**15)
      Q(K)=PHI(K)*OK(K,1)
6     CONTINUE
      CALL SOLVE
      CALL CORNER(IK);IF(IK.EQ.1)GOTO 102
100   CONTINUE
102   CALL WRITE
      STOP
      END
C*****************************************************************
      SUBROUTINE MAIN1(I,IT)
C*****************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/B1/PHI(434),Q(434),OK(434,9),NO(135),NNO(3)
      COMMON/B2/NH,NV,NN,NE,NVE,ILT,N,IROW,ICOL,DX,DY,XC,YC,BK(3,3),X(3)
     1,Y(3),Q1(3),B(3),DEL
D     TYPE 98
98    FORMAT(' ENTERS MAIN1')
      GOTO(1,2)IT
1     NNO(1)=I/2+IROW;NNO(3)=NNO(1)+NV;NNO(2)=NNO(3)+1
      RETURN
2     NNO(1)=I/2+IROW-1;NNO(2)=NNO(1)+1;NNO(3)=NNO(2)+NV
      RETURN
      END
C*****************************************************************
      SUBROUTINE MAIN2(I,IT,PHII,PHIF)
C*****************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/B1/PHI(434),Q(434),OK(434,9),NO(135),NNO(3)
      COMMON/B2/NH,NV,NN,NE,NVE,ILT,N,IROW,ICOL,DX,DY,XC,YC,BK(3,3),X(3)
     1,Y(3),Q1(3),B(3),DEL
D     TYPE 98
98    FORMAT(' ENTERS MAIN2')
      GOTO(1,2)IT
1     X(1)=FLOAT(IROW-1)*DY+PHII;X(2)=X(1)+DX;X(3)=X(2)
      Y(1)=PSIF-FLOAT(ICOL-1)*DY;Y(2)=Y(1)-DY;Y(3)=Y(1)
      RETURN
2     X(1)=FLOAT(IROW-1)*DX+PHII;X(2)=X(1);X(3)=X(1)+DX
      Y(1)=PSIF-FLOAT(ICOL-1)*DY;Y(2)=Y(1)-DY;Y(3)=Y(2)
      RETURN
      END
C*****************************************************************
      SUBROUTINE MAIN3(A,BT,C,AL)
C*****************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/B1/PHI(434),Q(434),OK(434,9),NO(135),NNO(3)
      COMMON/B2/NH,NV,NN,NE,NVE,ILT,N,IROW,ICOL,DX,DY,XC,YC,BK(3,3),X(3)
     1,Y(3),Q1(3),B(3),DEL
D     TYPE 98
```

```
98    FORMAT(' ENTERS PATH3')
      C2=EXP(2.*(A*((XC-X(2))*(Y(3)-Y(2))-(YC-Y(2))*(X(3)-X(2)))+
     1B*((YC-Y(1))*(X(2)-X(1))-(XC-X(1))*(Y(2)-Y(1)))+C*((XC-X(3))*(
     2Y(3)-Y(1))-(YC-Y(3))*(X(3)-X(1)))))
      A1=(1.+1.2*C2)/((1.-.2*C2)**6)
      RETURN
      END
C*************************************************************
      SUBROUTINE WRITE
C*************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/B1/PHI(434),Q(434),QK(434,9),NO(135),NNO(3)
      COMMON/B2/NH,NV,NN,NF,NVE,TLT,N,IROW,ICOL,DX,DY,XC,YC,BK(3,3),X(3)
     1,Y(3),Q1(3),B(3),DEL
      DO 10I=1,NN;BAB=EXP(Q(I));PHI(I)=(2.*BAB/(1.-BAB*BAB))/.80176
10    CONTINUE
      OPEN(UNIT=1,DEVICE='DSK',FILE='DESIGN.DAT',ACCESS='APPEND')
      WRITE(1,3)(PHI(I),I=1,NN)
3     FORMAT(7(F6.4,1X))
      CLOSE(UNIT=1,DEVICE='DSK',FILE='DESIGN.DAT',DISPOSE='SAVE')
      RETURN;END
C*************************************************************
      SUBROUTINE PLACE
C*************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/B1/PHI(434),Q(434),QK(434,9),NO(135),NNO(3)
      COMMON/B2/NH,NV,NN,NF,NVE,TLT,N,IROW,ICOL,DX,DY,XC,YC,BK(3,3),X(3)
     1,Y(3),Q1(3),B(3),DEL
D     TYPE 98
98    FORMAT(' ENTERS PLACE')
55    FORMAT(' ENTERS PLACE')
      DO 4 I=1,NV;NO(I)=I;PHI(I)=Q(I);Q(I)=0.0
4     CONTINUE;K1=1+NV;K2=1+NV*(NH-1);K3=NN;DO 5 I=K2,K3;NO(K1)=I;PHI(I)
     1=Q(K1);Q(K1)=0.0
5     K1=K1+1;K1=1+2*NV;K2=1+NV;K3=1+NV*(NH-2);DO 6 I=K2,K3,NV
      NO(K1)=I;PHI(I)=Q(K1);Q(K1)=0.0
6     K1=K1+1;K1=1+2*(NV-1)+NH;K2=2*NV;K3=NV*(NH-1);DO 7 I=K2,K3,NV
      NO(K1)=I;PHI(I)=Q(K1);Q(K1)=0.0
7     K1=K1+1;DO 8 M=1,NN;Q(M)=0.0
8     CONTINUE;RETURN;END
C*****************************************************************
      SUBROUTINE SOLVE
C*****************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/B1/PHI(434),Q(434),QK(434,9),NO(135),NNO(3)
      COMMON/B2/NH,NV,NN,NF,NVE,TLT,N,IROW,ICOL,DX,DY,XC,YC,BK(3,3),X(3)
     1,Y(3),Q1(3),B(3),DEL
D     TYPE 98
98    FORMAT(' ENTERS SOLVE')
D     PRINT 97,(Q(IQ),IQ=1,NN)
```

```
97    FORMAT(/(2X,F11.6))
C**********
C**********FORWARD PASS
C**********
      NN1=NN-1
        DO 20 L=1,NN1
      K1=0
        L1=L+1
        L2=ILT+L
        I1=2
        DO 20 I=L1,L2
      IF(I1.GT.ILT)GOTO 20
      IF(OK(L,I1).EQ.0.)GOTO 20
        G=-OK(L,I1)/OK(L,1)
      IP=2+K1
      Q(I)=Q(I)+G*Q(L)
99    FORMAT(' VALUE ERROR')
        DO 10 K=2,ILT
      J=K-1
      IF(IP.GT.ILT)GOTO 10
        OK(I,J)=OK(I,J)+G*OK(L,IP)
10      IP=IP+1
      K1=K1+1
20      I1=I1+1
C**********
C**********BACKWARD PASS
C**********
        DO 7 I=1,NN
        J=NN-I+1
        DO 6 K=2,ILT
        K1=J+K-1
        IF(K1.GT.NN)GOTO 5
        GOTO 8
5       BB=0.
        GOTO 6
8       BB=Q(K1)
6       Q(J)=Q(J)-OK(J,K)*BB
7       Q(J)=Q(J)/OK(J,1)
      RETURN
        END
C************************************************************************
      SUBROUTINE CONVER(IT)
C************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/B1/PHI(434),Q(434),OK(434,9),NO(135),NNO(3)
      COMMON/B2/NH,NV,NN,NF,NVE,ILT,N,IROW,ICOL,DX,DY,XC,YC,BK(3,3),X(3)
      1,Y(3),Q1(3),B(3),DEI
D     TYPE 98
98    FORMAT(' ENTERS CONVER')
      XP=10.**(-12);ADIF=0.0
```

```
      DO 10 I=1,NS;IF(Q(I).EQ.0.0)GOTO 10
      ADIF=ABS((Q(I)-PHI(I))/Q(I))+ADIF
      PHI(I)=REL*(Q(I)-PHI(I))+PHI(I)
10    CONTINUE
      TYPE *,ADIF
      IT=5;IF(ADIF.LE.XE)IT=4;RETURN;END
```

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